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WREATH PRODUCTS OF LIE ALGEBRAS

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1. Introduction

1.1. In 1964 A.I. Šmel'kin [12] introduced the notion of a verbal wreath product of groups. The analogous product for Lie algebras was introduced by Šmel'kin [11] in 1973. In particular he defined the verbal $\mathscr{B}$-wreath product W of the Lie algebras A and T where A is contained in a variety of Lie algebras over a commutative ring Λ with 1. This wreath product turns out to be a semidirect product of B and T, where B is the ideal generated by A.

In general it is not an easy task to decipher this wreath product. However, Šmel'kin remarks that this wreath product has a fairly simple description if $\mathscr{B}$ is the variety of abelian Lie algebras, and the underlying coefficient ring Λ is a field. In this case, B also turns out to be a free T-module whose rank is equal to the dimension of A. In this special situation W is simply denoted $A \operatorname{wr} T$.

In this paper we will be mainly concerned with such wreath products. However, we will adopt a rather different viewpoint towards wreath products. Thus we will define, in general, quite different wreath products from those introduced by Šmel'kin [11]. In the spirit of group theory, we will define wreath products in terms of generators and defining relations (see Section 3.1). Indeed, we are motivated throughout by group theory. In fact this paper may be viewed as an attempt to carry over to Lie algebras some theorems in group theory. The following section will specifically outline the theorems to be considered.

1.2. We begin in Section 3 with a discussion on the way in which our wreath products are constructed. Here, as in the case of Šmel'kin's Theorem, the bottom B is defined to be the least ideal of W generated by A. We first prove

Theorem 1. *If A is an abelian Lie algebra and T is an arbitrary Lie algebra, then the bottom B of $W = A \operatorname{wr} T$ is also abelian. If B is viewed as a right $U = U(T)$-module, then B is actually a free U-module freely generated by any chosen basis of the vector space A.*

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This theorem provides us with a nice way to visualize our wreath products.

We now turn our attention to proving Lie algebra analogs of some group theoretical results. In 1959 G. Baumslag [2] proved that if W is the wreath product of a nontrivial group A by an infinite group B, then W has trivial center. In Section 3.3 we prove the analogous result for Lie algebras. In particular we prove

Theorem 2. *If A is a nontrivial abelian Lie algebra and T is any nontrivial Lie algebra, then the center $W = A \operatorname{wr} T$ is trivial.*

In Section 4 we look at the analog of a theorem of P.M. Neumann. In 1964 P. Neumann [10] proved that the wreath product of a nontrivial group A by an infinite group B is directly indecomposable. In order to prove the Lie algebra analog (Theorem 4) we need the following theorem:

Theorem 3. *Let A be a nontrivial abelian Lie algebra and let T be any nontrivial Lie algebra. There exists a nontrivial element of $W = A \operatorname{wr} T$ whose centralizer is one-dimensional.*

This theorem together with some straightforward lemmas, easily yield

Theorem 4. *The wreath product of a nontrivial abelian Lie algebra A by any nontrivial Lie algebra T is directly indecomposable.*

Theorems 1 and 2 are not hard to prove; however, Theorem 3 is somewhat more difficult, requiring a fairly detailed examination of centralizers in certain universal enveloping algebras.

We prove next in Section 5 a simple, purely group-theoretical result, viz.,

Theorem 5. *Let A and T be two nontrivial groups. Then $W = A \operatorname{wr} T$ is freely indecomposable.*

The Lie algebra analog of Theorem 5 then also holds. In Section 6 we prove

Theorem 6. *The wreath product of a nontrivial abelian Lie algebra by any nontrivial Lie algebra is freely indecomposable.*

The proof of Theorem 6 makes use of the fact that the universal enveloping algebra of a free product of Lie algebras is a free product of associative algebras.

2. Preliminaries

2.1. Some basic notation, notions and definitions

Throughout this paper our Lie algebras will be taken over a commutative field F. Let A be a Lie algebra. As usual, for any $a, b \in A$, the product of a and b in A is denoted $[a, b]$. We denote the universal enveloping algebra of A by $U(A)$. (When there is no possibility of confusion we sometimes denote the universal enveloping algebra of a Lie algebra simply by U.) It is an easy consequence from the Poincaré–Birkhoff–Witt Theorem that U, an associative algebra, is a ring with no zero divisors. If X is a subset of a Lie algebra A, then $L\langle X \rangle$ denotes the Lie subalgebra of A generated by X. The center $Z(A)$ of A is defined by: $Z(A) = \{z \in A \mid [a, z] = 0$ for all $a \in A\}$. Let p be an element of a Lie algebra A. Then the centralizer of p, $\mathrm{cr}(p)$, is defined to be $\{x \in A \mid [x, p] = 0\}$. If X is a set, L is a Lie algebra and $X \subseteq L$, then by $\mathrm{id}_L(X)$ we mean the least ideal of L containing X.

If A and B are Lie algebras, then $A *_L B$ will denote the free product of the Lie algebras A and B. B is said to be the free product of a family of Lie algebras $\{H_j \mid j \in J\}$ if

(i) $H = L\langle \bigcup H_j \mid j \in J \rangle$.

(ii) For every Lie algebra H' and every family of homomorphisms $\alpha_j : H_j \to H'$, there exists a homomorphism α such that $\alpha : H \to H'$ extends the α_j $(j \in J)$.

Let A and B be Lie subalgebras of a Lie algebra W. To express this fact we write: $A \leq W$, $B \leq W$. W is said to be the *direct product* of its subalgebras A and B $(W = A \times B)$ if A and B are ideals of W $(A \trianglelefteq W, B \trianglelefteq W)$ and W is the direct sum of A and B $(W = A \oplus B)$.

A Lie algebra L is said to be *directly decomposable* if there exists nonzero subalgebras C and D of L such that $L = C \times D$. A Lie algebra L is said to be *freely decomposable* if there exists nonzero subalgebras C and D of L such that $L = C *_L D$.

2.2. Presentations

We shall view presentations of Lie algebras in the usual way. Specifically, if L is a Lie algebra and $(X; R)$ is a presentation, then we shall write $L = (X; R)$ if there exists a map $\theta : X \to L$ (set map) such that the extension ϕ of θ to $F = F(X) =$ the free Lie algebra on X satisfies: (i) ϕ is onto, and (ii) $\ker \phi = \mathrm{id}_F(R)$.

We will need the following analog of a well-known theorem from group theory (see e.g., P.M. Cohn [3, p. 153]).

von Dyck's Theorem. *Suppose A and B are Lie algebras where A and B can be presented as follows: $A = (X; R)$ and $B = (X; R \cup S)$. Then the map that sends x qua element of A to x qua element of B defines a homomorphism of A onto B.*

3. Wreath products

3.1. The definition

We recall the following definition from group theory. If A and T are two groups where $A=(X;R)$ and $T=(Y;S)$, then a presentation for the *wreath product* of A and T ($W=A \operatorname{wr} T$) is

$$A \operatorname{wr} T=(X\cup Y;\ R\cup S\cup\{[x,x'^{w(y)}]\mid x,x'\in X \text{ and } w(y) \text{ is any word in } Y \text{ not } 1 \text{ in } T\}).$$

(Here, $[u,v]$ denotes the commutator, $u^{-1}v^{-1}uv$, of u and v, and $u^v=v^{-1}uv$, as usual.) Analogously, if A and T are two Lie algebras where $A=(X;R)$ and $T=(Y;S)$, then we define the *wreath product* of A and T ($A \operatorname{wr} T$) by

$$A \operatorname{wr} T=(X\cup Y;\ R\cup S\cup\{[[w(x),v(y)],[w'(x),v'(y)]]\} \cup\{[w(x),[w'(x),v'(y)]]\}).$$

3.2. The wreath product of an abelian Lie algebra by an arbitrary Lie algebra

It is quite apparent that the structure of a Wreath product of Lie algebras given by the presentation above might well be difficult to decipher. In order to further discern the make-up of wreath products, we will need the following:

Theorem 1. *If A is an abelian Lie algebra and T is an arbitrary Lie algebra, then the bottom $B=\mathrm{id}_W(A)$ of $W=A \operatorname{wr} T$ is also abelian. If B is viewed as a right $U=U(T)$-module, then B is actually a free U-module freely generated by any chosen basis of the vector space A. Moreover, W is the semidirect product of B and T, denoted $W=B\rtimes T$.*

But we first make some remarks.

It can be easily deduced from von Dyck's Theorem that A and T are embedded in W. Notice that if M is an abelian ideal of a Lie algebra L, then M can be turned into an L/M- (Lie algebra) module by setting $m(a+M)=[m,a]$ ($m\in M, a\in L$). This 'action' of L/M on M can be extended to an action of $U(L/M)$ on M. This then turns M into a right $U(L/M)$-module in the usual sense.

The proof of Theorem 1 will depend in part on the following construction. The hypothesis of Theorem 1 will be adopted throughout.

Let $\bar{A}$ and $\bar{T}$ be copies of A and T respectively. Let $\bar{B}$ be the free right $U(\bar{T})$-module on a basis for $\bar{A}$. We now form the *semidirect product* $\bar{W}=\bar{B}\rtimes\bar{T}$ of $\bar{B}$ and $\bar{T}$ using this action of $U(\bar{T})$ on $\bar{B}$.

We remind the reader of the details. First the vector space structure of $\bar{W}$ is defined by setting $\bar{W}=\bar{T}\oplus\bar{B}$. Then $\bar{W}$ is turned into a Lie algebra by defining

$$[y_1+x_1,y_2+x_2]=[y_1,y_2]+(x_1y_2-x_2y_1) \tag{1}$$

where $x_i \in \bar{B}$, $y_i \in \bar{T}$ for $i=1,2$. Of course here $x_i y_j$ denotes the effect of $y_j \in \bar{T}$ on $x_i \in \bar{B}$. From these remarks, together with von Dyck's theorem, we may easily conclude that a canonical homomorphism θ, exists from W to $\bar{W}$, $\mathrm{id}_W(\bar{A}) = \bar{B}$ and θ restricted to B is an isomorphism from B to $\bar{B}$. Finally we have

Lemma 1. *$\theta: W \to \bar{W}$ is an isomorphism.*

Corollary 1.1. *If $W = A \operatorname{wr} T$ and if again A is abelian and $B = \mathrm{id}_W(A)$, then $W = B \oplus T$.*

The details of Lemma 1 and its Corollary are easily checked, and so are omitted. This concludes the proof of Theorem 1.

3.3. The proof of Theorem 2

We prove in this section the straightforward but pleasing

Theorem 2. *If A is a nontrivial abelian Lie algebra and T is any nontrivial Lie algebra, then the center of $W = A \operatorname{wr} T$ is trivial.*

We begin the proof of Theorem 2 with the following comments, assuming the hypothesis of Theorem 2 and Section 3.2 throughout. Consider the set of 2×2 matrices

$$N = \left\{ \begin{pmatrix} u & 0 \\ b & 0 \end{pmatrix} \middle| u \in U(T), b \in B \right\}.$$

These matrices can be multiplied naturally, and as such, form an associative algebra. For any $s, t \in N$ define $[s,t] = st - ts$. As usual, the vector space N with this operation [,] is a Lie algebra which we denote by N^-. Consider

$$M = \left\{ \begin{pmatrix} t & 0 \\ b & 0 \end{pmatrix} \middle| t \in T, b \in B \right\}.$$

Clearly $M \le N^-$. Define a vector space map $\phi: W \to M$ by

$$\phi: t + b \mapsto \begin{pmatrix} t & 0 \\ b & 0 \end{pmatrix}.$$

It is easy to prove that ϕ is a Lie algebra isomorphism.

Lemma 2. *The center $Z(M)$ of M is zero.*

Proof. In order to see this, one need only consider the product of an arbitrary element of $Z(M)$ with a particular element of M, viz., $\left(\begin{smallmatrix} 0 & 0 \\ b & 0 \end{smallmatrix}\right)$, along with the fact that U is a ring with no zero divisors.

Combining Lemma 2 with the fact that ϕ is an isomorphism it follows that $Z(M) = 0$.

4. Direct decompositions of wreath products of Lie algebras

4.1. It is not known what happens in the general case of direct decompositions of wreath products of Lie algebras because the structure of the wreath product of arbitrary Lie algebras is unclear. Hence, in this section, we consider $W = A \operatorname{wr} T$ where A, T, B, U and W are as in 3.2.

We begin with some necessary lemmas, definitions and remarks.

4.2. It is easy to prove, using (1), that W is not the direct product of B and some other ideal T' of W. Notice that if L is any Lie algebra such that $L = C \times D$, $C \leq L$, $D \leq L$, C and D are both nonzero, and if $f \in L$ then $\dim(\operatorname{cr}(f)) \geq 2$ (where dim denotes dimension). This follows directly from the fact that C and D centralize each other.

The concept of centralizers is crucial to the proof of Theorems 3 and 4. We return now to W and examine $w = a + t$ where a is a basis element of A and t is a nonzero element of T. If $c \in \operatorname{cr}(W)$, then $0 = [a, c] + [t, c]$. From this it easily follows that a nonzero element $c \in \operatorname{cr}(a + t)$ is neither an element of B nor an element of T.

The stage will be set for the proofs of Theorem 3 and 4 once we have made the following definitions. But first we need to recall the following information.

If $\{u_j \mid j \in J\}$ is an ordered basis for T, then the elements

$$u_{i_1} \cdot u_{i_2} \cdot u_{i_3} \cdots u_{i_r} \quad (i_1 \leq i_2 \leq \cdots \leq i_r)$$

and 1 form a basis for U (by the Poincaré–Birkhoff–Witt Theorem [3]). We term an element m of U a monomial if it is writen in the form

$$m = \alpha u_{i_1} u_{i_2} \cdots u_{i_r} \quad (\alpha \in F, \alpha \neq 0)$$

where the u's are basis elements of T. We allow $r = 0$ in which case m is interpreted simply as α. A monomial m of U is termed a *straight monomial* if $i_1 \leq i_2 \leq \cdots \leq i_r$. The weight of such a straight monomial is defined to be r. To express this fact we will write $\operatorname{wt}(m) = r$. (Notice that the definitions exclude zero from the straight monomials.) Finally an element $u \in U$ is termed *homogeneous* of weight r if it can be written as a sum of straight monomials of weight r. Notice that if $u \in U$ $(u \neq 0)$ then it can be written uniquely in the form

$$u = u(0) + u(1) + \cdots + u(p) \quad (u(p) \neq 0) \tag{2}$$

where $u(r)$ is homogeneous of weight r or is zero. Notice also that $u(p)$ is the term of highest weight.

4.3. The proof of Theorem 3

We adopt here the notation of 4.2; our objective is to prove the centralizer C of $w = a + t$ is one-dimensional, which suffices to prove Theorem 3. We will divide the proof of this assertion into two lemmas. The first is well known.

Lemma 3. *If* $\alpha u_{i_1}u_{i_2}\cdots u_{i_r}$ *is a straight monomial and if* u_k *is any basis element of* T*, then*

$$\alpha u_{i_1}u_{i_2}\cdots u_{i_r}\cdot u_{i_k}=\sigma+\alpha u_{i_1}\cdots u_j u_{i_k}u_{i_{j+1}}\cdots u_{i_r}$$

where σ *is either zero or a sum of straight monomials of weight less than* $r+1$ *and* $i_1\le i_2\le\cdots\le i_j\le k\le i_{j+1}\le\cdots\le i_r$.

It follows at once from Lemma 3 that we have

Lemma 4. *If* $u=u(0)+\cdots+u(p)$, $(u(p)\neq 0)$, *and* u_k *is a basis element of* T*, then*

$$u\cdot u_k=v(0)+v(1)+\cdots+v(p+1)\quad (v(p+1)\neq 0),$$

and $v(p+1)$ *is homogeneous of weight* $p+1$.

Corollary 4.1. *If* $u=u(0)+\cdots+u(p)$, $(u(p)\neq 0)$, *and* $t=\sum\alpha_j u_j\in T$, $t\neq 0$, *then*

$$u\cdot t=v(0)+v(1)+\cdots+v(p+1)\quad (v(p+1)\neq 0)$$

where $v(i)$ *is either zero or homogeneous of weight* i.

Before applying the lemmas above, we make use of the following calculation. Suppose $c\in C=\mathrm{cr}(a+t)$. We can write c as follows: $c=d+s$, $d\in B$, $s\in T$. We know from 4.2 that both d and s are nonzero. Consider the following:

$$[a+t,d+s]=[a,d]+[a,s]+[t,d]+[t,s]=0.$$

This implies that $as-dt=0=[s,t]$ since $B\cap T=0$. So $as=dt$ and $[s,t]=0$.

Now if $d=\sum a_i u_i$, $u_i\in U$, then $dt=\sum a_i(u_i t)$. Now $as=dt=\sum a_i(u_i t)$. Hence for some $k, a_k=a$, $s=u_k t$ and $u_i t=0$ $(i\neq k)$. Since U is a ring with no zero divisors it follows thence that $u_i=0$ for $i\neq k$ and that $d=au$, where $u=u_k$, and $s=ut$.

We need to examine $s=ut$ further. It should be noted that this equation involves only elements of T.

Since $s\in T$, $s=s(1)$ by (2). We now write u in the form $u=u(0)+\cdots+u(p)$, $(u(p)\neq 0, p\ge 0)$. So $ut=v(0)+\cdots+v(p+1)$ by Corollary 4.1. Hence, we obtain $s(1)=v(0)+\cdots+v(p+1)$. Hence $p=0$ is the only possibility, that is, $u\in F$. Therefore the centralizer of $w=a+t$ is one-dimensional as desired. This completes the proof of Theorem 3.

Combining the remarks on centralizers in 4.2 and Theorem 3, we immediately obtain:

Theorem 4. *The wreath product of a nontrivial abelian Lie algebra* A *by any nontrivial Lie algebra* T *is directly indecomposable.*

5. The proof of Theorem 5

We now prove

Theorem 5. *Let A and T be two nontrivial groups. Then $W=A \operatorname{wr} T$ is freely indecomposable.*

Proof. A subgroup of a free product is itself a free product of a free group and conjugates of subgroups of the factors. So if

$$W=A \operatorname{wr} T=X*Y \quad (A\neq 1\neq T, X\neq 1\neq Y),$$

then the base group B of W is itself a free product. Now by a theorem of Baer & Levi [1] a nontrivial free product is directly indecomposable. So B is itself a conjugate of a subgroup of X or Y or is free. But (by the same theorem of Baer & Levi) free groups are directly indecomposable. So B is a conjugate of a subgroup of X or Y. Since B is normal in W, this means that either $B\leq X$ or $B\leq Y$. But by the normal form theorem for free products, a nontrivial subgroup of a factor is not normal. This proves Theorem 5.

6. Free product descriptions of wreath products of Lie algebras

Theorem 6. *If $W=A \operatorname{wr} T$ is the wreath product of a nontrivial abelian Lie algebra A and an arbitrary Lie algebra T, then W is freely indecomposable.*

Theorem 6 will follow immediately from:

Lemma 5. *Suppose that the algebra P is a nontrivial free product*

$$P=C*_L D \quad (C\neq 0\neq D).$$

Then the only abelian ideal of L is zero.

Proof. Let U be the universal enveloping algebra of P. It is well known that $U=U(C)*_a(U(D)$ (where $U(C)*_a U(D)$ denotes the free product of associative algebras). In view of this, we will carry out all of our computations in U. We can then avail ourselves of the usual description of the elements of U in terms of bases for $U(C)$ and $U(D)$. We remind the reader of the details.

Let $\{x_i^C \mid i\in I\}$ be an ordered basis for C. Then 1 together with

$$x_{i_1}^C x_{i_2}^C \cdots x_{i_m}^C, \quad i_1\leq\cdots\leq i_m \tag{3}$$

constitute a basis for $U(C)$. Similarly, if $\{x_j^D \mid j\in J\}$ is an ordered basis for D, 1 together with

$$x_{j_1}^D x_{j_2}^D \cdots x_{j_s}^D, \quad j_1\leq\cdots\leq j_s \tag{4}$$

constitute a basis for $U(D)$. A basis for the associative algebra $U=U(C) *_a U(D)$ consists of 1 and the set of monomials

$$z=w_{k_1}^{\alpha_1} w_{k_2}^{\alpha_2} \cdots w_{k_t}^{\alpha_t} \tag{5}$$

where $\alpha_i \in \{C, D\}$, $\alpha_i \neq \alpha_{i+1}$ and w_{k_i} is of the form (3) if $\alpha_i=C$ and of the form (4) if $\alpha_i=D$.

Let X be an ordered basis for C and let Y be an ordered basis for D. Let B be a nontrivial abelian ideal of P. Notice that if either $x \in C$ or $y \in D$, then $[b,[b,x]]=0=[b,[b,y]]$ for every $b \in B$. In particular, choose some $x \in X$ and $b \neq 0$, then from

$$0=[b,[b,x]] \tag{6}$$

we obtain

$$b^2x=(2bx-xb)b. \tag{7}$$

In view of the description in (5) for $b \in U(C) *_a U(D)$, we may write b uniquely in the form

$$b=\sum \gamma w_1 \cdots w_r \tag{8}$$

where γ is a nonzero element of the underlying ground field F. It follows that

$$b=cx+\sum \delta dx'+\sum \sigma ey \tag{9}$$

where x is the element of X chosen in (6), $x' \in X-\{x\}$, and $y \in Y$.

We assume that $\sum \sigma ey \neq 0$. Notice that if this were not true, the expression $[b,[b,x]]$ in (6) could be replaced by $[b,[b,y]]$, and the argument would proceed in the same manner.

Substituting (9) in the right side of (7) we may write the right side of (7) as follows:

$$\begin{aligned}&(2bx-xb)(cx+\sum \delta dx'+\sum \sigma ey)\\ &\quad=(2bx-xb)(cx)+(2bx-xb)(\sum \delta dx')+(2bx-xb)(\sum \sigma ey).\end{aligned} \tag{10}$$

Consider the following term from the right side of (10): $(2bx-xb)(\sum \sigma ey)$. Clearly all of these terms will end in some $y \in Y$. So from (7) we must have $(2bx-xb)(\sum \sigma ey)=0$. Since U is a ring with no zero divisors, $(2bx-xb)=0$ and from (7) we must have that $b^2x=0$. Since $x \neq 0$, it follows that $b=0$. Hence $B=0$ as desired.

References

[1] R. Baer and F. Levi, Freie Produkte und ihre Untergruppen, Compositio Math. 1 (1936) 391–398.

[2] G. Baumslag, Wreath products and p-groups, Proc. Cambr. Phil. Soc. 55 (1959) 224–231.

[3] P.M. Cohn, Universal Algebra (Reidel, Boston, 1981) 153. (Originally published in 1965, Harper & Row.)

[4] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory (Springer, New York, 1972).

[5] N. Jacobson, Lie Algebras (Wiley Interscience, New York, 1962).
[6] A.G. Kurosh, The Theory of Groups, I and II (Chelsea, New York, 1956).
[7] W. Magnus, A. Karrass and D. Solitar, Combinatorial Group Theory (Wiley Interscience, New York, 1966).
[8] A.I. Mal'cev, Generalized nilpotent algebras, AMS Transl., Series 2 (1969) 347-366.
[9] H. Neumann, Varieties of Groups, Ergebnisse der Math. und ihrer Grenzgebiete, Band 37 (Springer, New York, 1967).
[10] P.M. Newman, On the structure of standard wreath products of groups, Math. Z. 84 (1964) 343-373.
[11] A.I. Šmel'kin, Wreath products of Lie algebras and their applications to the theory of groups, Trans. Moscow Math. Soc. 29 (1973) 239-252.
[12] A.I. Šmel'kin, Wreath products and varieties of groups, Dokl. Akad. Nauk. SSSR 157 (1964).